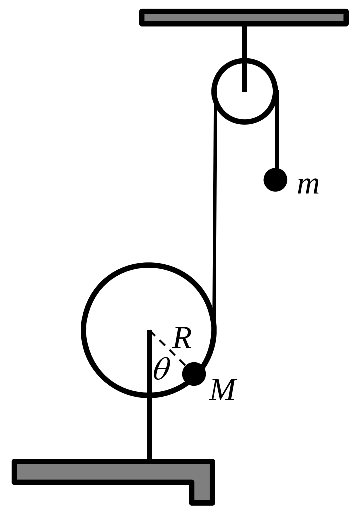
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 13, 2015, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines, 'Mathematical Handbook of Formulas and Tables'.

A1. A mass M is attached to a massless hoop of radius R that lies in a vertical plane. The hoop is free to rotate about its fixed center. The mass M is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string. Find the equation of motion for the angle of rotation of the hoop θ. For what value of θ does equilibrium occur? What is the frequency of small oscillations about equilibrium? Assume that m moves only vertically, and that $M > m$. **Note:** A convenient reference point for calculating gravitational potential energy is when mass M is at the bottom of the hoop.



A2. An uncharged metal (conducting) sphere of radius a is coated with a thick insulating shell (dielectric constant κ) out to radius b. This object is now placed in an otherwise uniform electric field, $\vec{E} = E_0\hat{z}$. Recall that if there is no ϕ dependence, the general solution to Laplace's equation for the potential is given as: $V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$.

a) Determine the potential $V(r,\theta)$ inside the thick insulating shell.

b) Determine the electric field $\vec{E}(r,\theta)$ inside the thick insulating shell.

A3. The antiproton $\overline{p}$ was discovered in the following reaction: $p + p \to p + p + p + \overline{p}$, in which accelerated protons were incident on a target of protons at rest in the laboratory frame. The threshold kinetic energy for this reaction to occur is defined as the minimum incident energy that will leave all of the particles moving together as a single unit, *i.e.,* all with the same final velocity vector. In the final center of mass reference frame, this unit would be stationary. Find the threshold kinetic energy to produce an antiproton in this reaction. Express your answer in multiples of the rest mass energy of the proton, mc^2, where m is the mass of the proton and the antiproton.

A4. A particle moves in a potential which has only four bound states. These bound states include a doubly degenerate ground state ε_0 corresponding to an orthonormal pair of states $|1\rangle$ and $|2\rangle$, and two non-degenerate excited states $|3\rangle$ and $|4\rangle$ with energies $3\,\varepsilon_0$ and $4\,\varepsilon_0$, respectively. A weak electric field E_0 is applied to the system. The matrix elements of the perturbing potential all vanish except for the following:

$$\langle 1|\,H_1\,|2\rangle = \langle 2|\,H_1\,|1\rangle = eE_0\,a_0$$

$$\langle 3|\,H_1\,|4\rangle = \langle 4|\,H_1\,|3\rangle = -3\,eE_0\,a_0$$

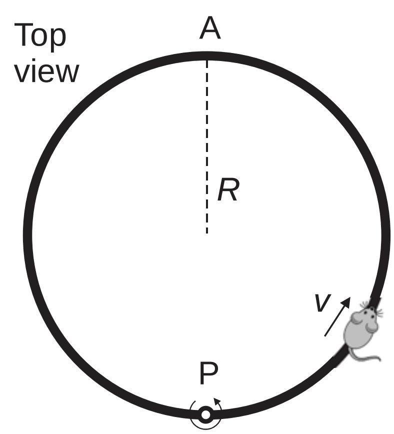
where $eE_0\,a_0 \ll \varepsilon_0$. In this last relation, e is the electronic charge and a_0 is a constant having units of length characteristic of the size of the system.

(a) Construct the matrices that represent H_0 and H_1 in this representation.

(b) **Use perturbation theory** to find the eigenvalues of the total Hamiltonian $H = H_0 + H_1$, to lowest non-vanishing order in the field. Qualitatively plot the energies of the system as a function of field strength, clearly showing the limiting behavior at small fields.

(c) Find the ground state of the system in the presence of the perturbing field.

A5. Consider a thin ring of radius R and mass M lying on a horizontal frictionless table. The ring is pivoted at point P; it can rotate freely about this point. A mouse of mass m is running around the ring with constant speed v with respect to the ring. The ring is at rest when the mouse passes the pivot point.



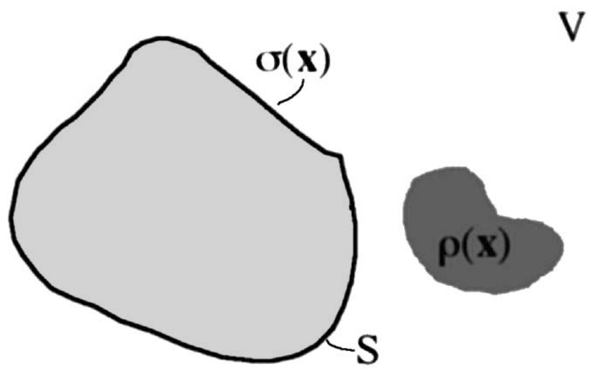
a) Find the angular speed ω of the ring when the mouse is at point A exactly opposite the pivot point.
b) Derive an expression for the angular speed ω of the ring as a function of the mouse's position along the ring.

Hint: You can treat part a) as a special case of b) but it may be easier to think about the simple geometry of part a) first.

A6. Consider the 3–dimensional region of space V ***outside*** a conducting surface S, as shown in the figure below. Let $\Phi(x)$ be the potential in V due to a charge density $\rho(x)$ in V and a surface charge density $\sigma(x)$ on the conductor. Likewise, let $\Phi'(x)$ refer to the potential in an unrelated problem in the same volume, with charge densities $\rho'(x)$ and $\sigma'(x)$. Note: $\sigma'(x)$ is defined on the same surface as $\sigma(x)$.

a) Use Green's Theorem to show that

$$\int \rho\Phi' dV + \int \sigma\Phi' dS = \int \rho'\Phi dV + \int \sigma'\Phi dS$$



b) A point charge q is a distance r away from the center of a ***grounded*** conducting sphere of radius R. Use the result in part a) above to find the net charge on the sphere. [Hint: take as the primed problem a sphere at uniform potential and without charge outside]

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 14, 2015, 1:00 – 5:00 p.m.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines', 'Mathematical Handbook of Formulas and Tables'.

B1. A particle of mass m moves in one dimension under the influence of a force $F(x,t) = \frac{k}{x^2}\exp(-t/\tau)$, where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

B2. Consider a one-dimensional potential well $V(x) = -aV_0\delta(x)$ where $\delta(x)$ is a delta function and a and V_0 are positive constants.

(a) Find the eigenfunction and energy eigenvalue of a point particle of mass m bound by this potential.

(b) Now consider a symmetric potential, $V(x) = -aV_0[\delta(x+a) + \delta(x-a)]$. Treating the degenerate bound states centered at each well in the absence of the other as the zeroth order unperturbed eigenstates, find the new bound state energy eigenfunctions and energy eigenvalues for this system.

(c) Are these solutions exact? If not, explain why; and discuss under what conditions (*i.e.,* with an appropriate inequality) such an approximate treatment is valid.

B3. Use the variational principle to estimate the ground-state energy of a particle in the potential

$V = \infty$ for $x < 0$ and $V = cx$ for $x > 0$, where c is a positive constant.

Take xe^{-ax} as the trial wave function for the ground state.

B4. A positive charge q is fired head-on at a distant positive charge Q with an initial velocity v_0. The charge Q is held stationary. The charge q of mass m comes in, decelerates to $v=0$, and returns out to infinity. Assume the motion is non-relativistic so that $v_0 \ll c$ and thus the power radiated is described by the Larmor formula $P=\frac{\mu_0}{6\pi c}q^2a^2$, where a is the acceleration of the charge q. Since the energy radiated is very small, you can ignore the effect of radiative losses on the motion of the particle. Determine the fraction of the initial energy $\left(\frac{1}{2}mv_0^2\right)$ that is radiated away. You may find it easier to start from the distance of closest approach and calculate the energy radiated as the charge returns to infinity and then multiply that by two.

B5. Consider a set of N atoms arranged on a circular ring, with equal spacing between them (as in a benzene ring). An electron in this system can reside on any of the N atoms, with an energy ε_0, but is able to move from the atom n on which it is sitting to either of its two neighbors at $n\pm1$. The corresponding Hamiltonian can be written

$$H=\sum_n |n\rangle\varepsilon_0\langle n|+\sum_n |n+1\rangle J\langle n|+\sum_n |n-1\rangle J\langle n|,$$

where it is assumed that the states $|n\rangle=|n+N\rangle$ are orthonormal and J is a real constant. Assume a solution to the energy eigenvalue equation of the form $\phi_n=\langle n|\phi\rangle=A\lambda^n$, and impose periodic boundary conditions $[\phi_n=\phi_{n+N}]$, to determine allowed values of λ. Use this to find the energy spectrum and to construct an orthonormal set of energy eigenstates for this Hamiltonian.

B6. Consider a classical ideal gas consisting of N identical particles of mass m in a cubic volume of linear size L. The gas is in thermal equilibrium at temperature T. A small square metal plate of linear size a (with $a \ll L$) is introduced into the gas.

a) Determine the average number of particles hitting one side of the plate during a short time interval Δt.

b) Particles hitting the plate with normal velocity less than v_0 are reflected while particles hitting with normal velocity larger than v_0 are absorbed by the plate. Find the average number of particles absorbed by one side of the plate during a short time interval Δt.